Closed Form Expressions for Brand Performance Measures

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Rungie and Goodhardt (2004) provide theoretical expressions for various brand performance measures. These expressions are critical to the further development of a model used in marketing to provide the distribution for the repeated purchases. However, at least five of the brand performance measures given by Rungie and Goodhardt (2004) are expressed as infinite sums. Here, we show that one can actually derive explicit expressions for the five brand performance measures.

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Introduction

The recent paper by Rungie and Goodhardt (2004) provides theoretical expressions for various brand performance measures, including the average purchase rate, market share, penetration, purchase frequency, 100% loyals, share of category requirements, average portfolios size, repeat rate and the category polarization. These expressions are critical to the further development of a model used in marketing to provide the distribution for the repeated purchases, by shoppers over a period of time, of the competing brands within a product category. Rungie and Goodhardt (2004) also provide computer programs to calculate the theoretical expressions.

At least five of the brand performance measures given by Rungie and Goodhardt (2004) are expressed as infinite sums. Rungie and Goodhardt (2004) state that "These can be numerically approximated by finite summations ...." Here, we show that one can actually derive explicit expressions for the five brand performance measures. Firstly, consider the penetration for brand j given by

$$\textit{Penetration for brand } j = \sum_{k=1}^{\infty} f(k) \left\{ 1 - \frac{\Gamma(S)\Gamma(S-\alpha_j+k)}{\Gamma(S+k)\Gamma(S-\alpha_j)} \right\} \quad (1)$$

where *f(k)* is the probability mass function (pmf) of the negative binomial distribution given by

$$f(k) = \frac{\Gamma(\gamma+k)\beta^k}{\Gamma(\gamma)k!(1+\beta)^{\gamma+k}} \quad (2)$$

for $k = 0,1,\dots$ (see Johnson *et al.* (1992)).

Because (2) is a pmf, one has

$$\sum_{k=1}^{\infty} f(k) = 1-(1+\beta)^{-\gamma}. \quad (3)$$

Also, using the Gauss hypergeometric function defined by

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} $$

(where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial), one can express

$$ \begin{aligned} & \sum_{k=1}^{\infty} f(k) \left\{ 1 - \frac{\Gamma(S)\Gamma(S-\alpha_j+k)}{\Gamma(S+k)\Gamma(S-\alpha_j)} \right\} \\ = \; & \frac{\Gamma(S)(1+\beta)^{-\gamma}}{\Gamma(\gamma)\Gamma(S-\alpha_j)} \sum_{k=1}^{\infty} \frac{\Gamma(\gamma+k)\Gamma(S-\alpha_j+k)}{\Gamma(S+k)k!} \left(\frac{\beta}{1+\beta} \right)^k \\ = \; & (1+\beta)^{-\gamma} \sum_{k=1}^{\infty} \frac{(\gamma)_k (S-\alpha_j)_k}{(S)_k k!} \left(\frac{\beta}{1+\beta} \right)^k \\ = \; & (1+\beta)^{-\gamma} \left\{ {}_2F_1\left(\gamma, S-\alpha_j; S; \frac{\beta}{1+\beta} \right) - 1 \right\}. \end{aligned} \tag{4} $$

Combining (1), (3) and (4), one obtains the following explicit expression:

$$ \textit{Penetration for brand } j = 1 - (1+\beta)^{-\gamma} \left\{ {}_2F_1\left(\gamma, S-\alpha_j; S; \frac{\beta}{1+\beta} \right) - 1 \right\}. \tag{5} $$

Secondly, using (5) and the facts

$$ \textit{Average purchase rate for brand } j = \frac{\alpha_j \beta \gamma}{S} $$

and

$$ \textit{Penetration for brand } j = \frac{\textit{Average purchase rate for brand } j}{\textit{Penetration for brand } j}, $$

one can obtain the explicit expression:

$$ \begin{aligned} & \textit{Purchase frequency for brand } j \\ = \; & \frac{\alpha_j \beta \gamma}{S} \left\{ 1 - (1+\beta)^{-\gamma} \, {}_2F_1\left(\gamma, S-\alpha_j; S; \frac{\beta}{1+\beta} \right) \right\}^{-1}. \end{aligned} \tag{6} $$

Thirdly, consider the 100% Loyals for brand j given by

$$\begin{aligned}&100\% \quad Loyals \quad for \quad brand \quad j\\ &= \frac{1}{Penetration \quad for \quad brand \quad j}\sum_{k=1}^{\infty} f(k)\frac{\Gamma(S)\Gamma(\alpha_j + k)}{\Gamma(S+k)\Gamma(\alpha_j)}.\end{aligned} \qquad (7)$$

Following arguments similar to those leading to (4), one can simplify (7) to the form:

$$\begin{aligned}&100\% \quad Loyals \quad for \quad brand \quad j\\ &= \frac{(1+\beta)^{-\gamma}}{Penetration \quad for \quad brand \quad j}\left\{ {}_2F_1\left(\gamma, S-\alpha_j; S; \frac{\beta}{1+\beta}\right) - 1\right\}.\end{aligned} \qquad (8)$$

Fourthly, consider the share of category requirements for brand j given by

$$\frac{Share \quad of \quad category \quad for \quad brand \quad j}{Average \quad purchase \quad rate \quad for \quad brand \quad j} = \left[\sum_{k=1}^{\infty} kf(k)\left\{1 - \frac{\Gamma(S)\Gamma(S-\alpha_j+k)}{\Gamma(S+k)\Gamma(S-\alpha_j)}\right\}\right]^{-1}. \qquad (9)$$

Because the mean of the pmf, (2), is $\beta\gamma$, one has

$$\sum_{k=1}^{\infty} kf(k) = \beta\gamma. \qquad (10)$$

Also,

$$\begin{aligned}&\sum_{k=1}^{\infty} kf(k)\left\{1 - \frac{\Gamma(S)\Gamma(S-\alpha_j+k)}{\Gamma(S+k)\Gamma(S-\alpha_j)}\right\}\\ &= \frac{\Gamma(S)\beta}{\Gamma(\gamma)\Gamma(S-\alpha_j)(1+\beta)^{\gamma+1}}\sum_{k=1}^{\infty}\frac{\Gamma(\gamma+1+k)\Gamma(S-\alpha_j+1+k)}{\Gamma(S+1+k)k!}\left(\frac{\beta}{1+\beta}\right)^k\\ &= \frac{\beta\gamma(S-\alpha_j)}{S(1+\beta)^{\gamma+1}}\sum_{k=1}^{\infty}\frac{(\gamma+1)_k\,\Gamma(S-\alpha_j+1)_k}{(S+1)_k\,k!}\left(\frac{\beta}{1+\beta}\right)^k\\ &= \frac{\beta\gamma(S-\alpha_j)}{S(1+\beta)^{\gamma+1}}\,{}_2F_1\left(\gamma+1, S-\alpha_j+1; S+1; \frac{\beta}{1+\beta}\right).\end{aligned} \qquad (11)$$

Combining (9), (10) and (11), one obtains the following explicit expression:

$$\begin{aligned}&\frac{Share \quad of \quad category \quad requirements \quad for \quad brand \quad j}{Average \quad purchase \quad rate \quad for \quad brand \quad j}\\ &= (\beta\gamma)^{-1}\left\{1 - \frac{S-\alpha_j}{S(1+\beta)^{\gamma+1}}\,{}_2F_1\left(\gamma+1, S-\alpha_j+1; S+1; \frac{\beta}{1+\beta}\right)\right\}^{-1}.\end{aligned} \qquad (12)$$

Finally, using (5) and the fact

$$Average\quad portfolios\quad size = \frac{\sum_{j=1}^{h} Penetration\quad for\quad brand\quad j}{Penetration\quad for\quad the\quad category},$$

one can obtain the explicit expression:

$$\begin{aligned} & Average\quad portfolios\quad size \\ = & \frac{1}{Penetration\quad for\quad the\quad category}\left\{h - (1+\beta)^{-\gamma}\sum_{j=1}^{h} F_1\left(\gamma, S-\alpha_j; S; \frac{\beta}{1+\beta}\right)\right\}. \end{aligned} \tag{13}$$

Note that all of the expressions in (5), (6), (8), (12) and (13) involve just the Gauss hypergeometric function. This function is quite well known and well established in the mathematics and statistics literature (see Sections 9.10–9.13 of Gradshteyn and Ryzhik (2000) for detailed properties). Numerical routines for its direct and exact computation are available in most computer packages (CAT, Maple, Mathematica, Matlab, R, Splus, etc).

References

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